Exotic Options

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1. Introduction

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1.1 Goal

The goal of this report is to give answers to four following questions:

- What is an Exotic Option?
- What sorts of Exotics are traded?
- Why are these options attractive?
- How to price several chosen Exotic Options?

1.2 Definition of Exotic Option

An Exotic Option is a more complex contract than simple European or American call or put option on stock, index, foreign currency, commodity or interest rate. Exotic options are the second generation of options. They have key terms different from or additional to those found in Vanilla (non-exotic) options.

1.3 Types of Exotic Options

Exotic options include:

- Quanto options
- Forward start options
- Compound options
- Chooser options
- Barrier options
- Binary options
- Lookback options
- Shout options
- Asian options
- Options to exchange one asset for another
- Options involving several assets

1.4 Why to write Exotic Options?

Exotic options offer the writer the opportunity to explore wider bid-offer spread. This is due to the fact that not many financial institutions trade exotics and the competition on the market is not as strong as on Vanilla option market. This fact makes also possible for the writer to maintain higher profit margin. Another good reason for trading exotic options is that they are considered as a sophisticated extension to Vanilla options.

1.5 Why to buy Exotic Options?

The main reason for buying exotic options is that they offer a tailor-made protection for a moderate price. A trader that has a view of declining volatility can employ a barrier option instead of strategy based on vanilla options since this solution is less expensive.

Exotic options also offer structured protection when Vanilla options can't be successfully employed. Consider a company that has revenues in many foreign currencies. This company is exposed to exchange rate risk in many currencies. The profits of this company can be protected against large movements of exchange rates be a very costly "strategy" of buying put options on each of these currencies. Instead, an exotic option on a basket of foreign currencies might be considered. This exotic option offers protection against large movements of the whole basket of currencies which actually is the case here because the profits of company under investigation depend on a join behavior of all the currencies.

1.6 Problems Concerning Exotic Options

Although exotic options have substantial advantages, they also have drawbacks. One of them is low liquidity on some exotic option markets. This might make difficult or even impossible to buy or sell sufficient amount of exotic options to hedge investor's portfolio. This might influence the marking-to-market process when rehedges have to be done. Another disadvantage of exotic options is that underlying market might become manipulative if large amounts of exotic options are traded and approach maturity. In some cases the writer might for instance try to "kill" the barrier options on less liquid underlying market if this would protect him against large loss when option expires in-the-money.

1.7 Options Under Thorough Investigation

In this report three exotic options will be under thorough investigation:

- Double Barrier Option - Call-up-and-out-down-and-out
- Arithmetic Average-Rate Asian Call Option
- Interest Rate Collar

2. Double Barrier Call Option

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2.1 Introduction

Barrier options are one of the most widely-traded exotics on some markets. When compared to vanilla options, they have one additional key term: a barrier imposed on price of underlying. The barrier might be below the strike or above the strike. When barrier is hit during the life of the option, an event occurs. There are two kinds of such events: the contract might be cancelled or the contract might become effective. Hence there are four basic types of barrier options:

- **down-and-out** – the barrier is below the strike price; once it is hit, the option "dies" and at maturity there is no payout for option holder, although the option might be in-the-money at maturity;
- **down-and-in** – the barrier is also below the strike price, but this time the contract becomes alive when the barrier is hit – at maturity option holder gets usual payout if and only if the barrier is hit during the life of the option; if the barrier is not hit, the option expires worthless, although it might be in-the-money at maturity;
- **up-and-out** – this time the barrier is above the strike price; once it is hit, the contract becomes "nulled" – there is no payout for option holder at maturity, no matter if the option is in-the-money or out-of-the-money;
- **up-and-in** – the barrier is also above the strike price; option holder gets usual payout if and only if the stock price hits the barrier during the life of the option.

These four basic features of barrier options apply to both call and put options, as well as to European and American options. The underlying of a barrier option might be a stock, stock index, commodity, foreign currency or interest rate.

Barrier options are clearly path-dependent. The payout at maturity depends not only on the price of underlying at maturity, but also on the way the price of underlying got to this particular level.

There are other possible features that can be applied to four basic types of barrier options. The option might be said to have two barriers instead of one. There are two basic types of double barrier options:

- up-and-put-down-and-out
- up-and-in-down-and-in

For the former type the contract “dies” if either lower or upper barrier is hit during the life of the option. For the latter type the option becomes effective if and only if either lower or upper barrier is hit.

Another feature that occurs with barrier options is a rebate. Barrier option with a rebate pays certain amount of money to option holder once the barrier is reached. This is often the case for “out” options – although the contract “dies”, the option holder receives rebate to reduce his loss on losing the payout. The rebate might be paid as soon as the barrier is reached or later, even at maturity.

Since double barrier options mentioned above are becoming more and more popular, they are not considered to be complicated any more. For investors keep seeking for features that make the contract complicated, there have been introduced double barrier options with “repeating hitting” feature. In this case option is cancelled (or becomes alive) once both barriers are hit before maturity.

Rainbow barrier options are another example of more complicated barrier options. Additional feature in this case is such that the barrier is imposed on one underlying, but the payout at maturity is calculated on the basis of another underlying.

Soft barrier options allow the contract to be gradually knocked-out or knocked-in. The barrier in such a case is split into two levels. Consider as an example up-and-out option. Once the lower part of the barrier is reached, the contracts starts to become worthless, but it is done proportionally as long as the upper part of the barrier is not reached. If, on one hand, the maximum price of the underlying lies somewhere between lower and upper part of the barrier, the option is knocked-out proportionally to how deep the price went into the interval. If, on the other hand, the maximum price is above upper part of the barrier, the contract is knocked-out in 100%.

The last type of barrier options which are worth mentioning are Parisian options. The difference between plain barrier options and Parisian options lies in for how long the underlying price is above (for “up” options) or below (for “down” options) the barrier. In case of plain barrier options it is sufficient that the barrier is reached. In case of Parisian options the price should stay beyond the barrier for a specified in advance time. On one hand this feature makes the contract less tractable to manipulation of underlying price close to the barrier and also makes the dynamic hedging easier. On the other hand pricing of Parisian options is far more complicated due to higher dimensionality of the problem.

2.2 Advantages of barrier options

The most frequently mentioned advantage of barrier options is that they are cheaper when compared to vanilla options with the same strikes and maturities. This is due to the fact that barrier option can never perform better the vanilla option, unless rebate paid to option holder when the barrier is hit is very large. In fact barrier options holder retains much more risk of future underlying price behavior than vanilla option holder.

Another advantage of barrier options is their flexibility in terms of setting the level of the barrier and thus the cost of the contract. The closer the barrier is to the strike, the higher is the probability that the barrier is reached and the lower is the price of the contract (for “out” options). This makes it possible to adjust the terms of barrier option to meet every particular investor’s requirements.

The barrier options can be written on every underlying – stocks, stocks indices, commodities, interest rates and (especially popular) foreign currencies.

2.3 Disadvantages of barrier options

It is not a surprise that barrier options have also shortcomings.

One of the most importance drawback is that the delta of the “out” options is very unstable when underlying price is close to the barrier. In fact, delta can easily become negative when underlying approaches the barrier. Gamma is also very large and negative close to the barrier. This fact makes delta hedging very difficult as well as very costly, especially when the investor has to buy the underlying instead of selling it.

Another disadvantage is that the contract can be “killed” on purpose by the writer close to maturity by making the underlying price reach the barrier for a very short time.

2.4 Call-up-and-out-down-and-out pricing formula

Barrier option that will be examined in details in our report is double barrier call. The contract has two barriers: one upper and one lower to the strike price. Both barriers are “out” barriers – once a barrier is reached, the option “dies”. Although it is easy to introduce barriers dependent on time, to make the example easier both barriers are flat.

Basic calculations are made for an at-the-money option on non-dividend paying stock currently pricing 100 dollars. Barriers are set to \$80 and \$130. The time to maturity is one quarter of a year (T=0.25). The risk free interest rate is assumed to be 10%. There are no costs of carry. The volatility of stock price is assumed to be 40% annually. No rebate is paid to option holder when a barrier is reached.

The value of the option is determined by the formula [see Haug (1998)]:

$$c = Se^{(b-r)T} \sum_{n=-\infty}^{+\infty} \left\{ \left(\frac{U^n}{L^n} \right)^{\mu_1} \left(\frac{L}{S} \right)^{\mu_2} \left[N(d_1) - N(d_2) \right] - \left(\frac{L^{n+1}}{U^n S} \right)^{\mu_3} \left[N(d_3) - N(d_4) \right] \right\}$$

$$- Xe^{-rT} \sum_{n=-\infty}^{+\infty} \left\{ \left(\frac{U^n}{L^n} \right)^{\mu_1 - 2} \left(\frac{L}{S} \right)^{\mu_2} \left[N\left(d_1 - \sigma\sqrt{T}\right) - N\left(d_2 - \sigma\sqrt{T}\right) \right] - \left(\frac{L^{n+1}}{U^n S} \right)^{\mu_3 - 2} \left[N\left(d_3 - \sigma\sqrt{T}\right) - N\left(d_4 - \sigma\sqrt{T}\right) \right] \right\}$$

where:

$$d_1 = \frac{\ln\left(SU^{2n} / XL^{2n}\right) + \left(b + \sigma^2/2\right)T}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\ln\left(SU^{2n} / FL^{2n}\right) + \left(b + \sigma^2/2\right)T}{\sigma\sqrt{T}}$$

$$d_3 = \frac{\ln\left(L^{2n+2} / XSU^{2n}\right) + \left(b + \sigma^2/2\right)T}{\sigma\sqrt{T}}$$

$$d_4 = \frac{\ln\left(L^{2n+2} / FSU^{2n}\right) + \left(b + \sigma^2/2\right)T}{\sigma\sqrt{T}}$$

$$\mu_1 = \frac{2\left[b - \delta_2 - n(\delta_1 - \delta_2)\right]}{\sigma^2} + 1$$

$$\mu_2 = 2n \frac{\delta_1 - \delta_2}{\sigma^2}$$

$$\mu_3 = \frac{2\left[b - \delta_2 + n(\delta_1 - \delta_2)\right]}{\sigma^2} + 1$$

$$F = Ue^{\delta_1 T}$$

and S is current underlying price, X is a strike price, U is upper barrier, L is lower barrier, r is risk-free interest rate, b is cost of carry, T is time to maturity (in years), σ is annualized volatility, $N(.)$ denotes cumulative distribution function of standardized Normal distribution and δ_1, δ_2 determine the curvature of the lower barrier L and the upper barrier U such that:

- $\delta_1 = \delta_2 = 0$ corresponds to two flat barriers;
- $\delta_1 < 0 < \delta_2$ corresponds to a lower barrier exponentially growing with time, while the upper barrier will be exponentially decreasing;
- $\delta_1 > 0 > \delta_2$ corresponds to convex downward lower barrier and convex upward upper barrier.

The value of the double barrier call option is expressed as an infinite series of weighted normal distribution functions. However, the convergence of the formula is quite rapid. Haug (1998) suggests that in most cases it is sufficient to calculate only two of three leading terms. In our calculations n from –10 to 10 was used.

Important issue concerning pricing of barrier options is how often is the underlying price monitored. Presented model assumes that the price is monitored continuously. However, this is not the case in practice. In practice the underlying price might be monitored hourly, daily, weekly or even monthly. In this report it is assumed that the underlying price is monitored **daily**.

2.5 Call-up-and-out-down-and-out properties

Using the formula shown above, the value of call-up-and-out-down-and-out contract assuming daily monitoring is **$3.15**. If continuous monitoring is employed, the value of the option drops to **$2.82** due to higher probability of reaching either upper or lower barrier. Corresponding plain vanilla call option (which can be thought of as an double barrier option with infinite interval between points when the price is monitored) has a value of **$7.77**.

Figure 1. shows the value of double barrier call option as a function of stock price at time T=0.25. Between the barriers the contract has a positive value with a peak value of $3.50 at $107. Below the lower barrier as well as above the upper barrier the option value is zero.

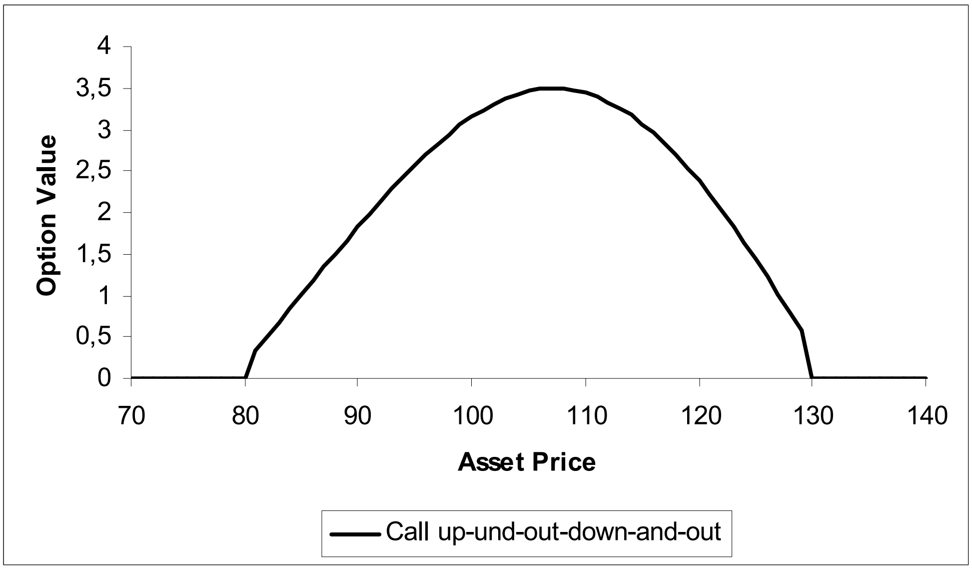

Fig. 1. The value of call-up-and-out-down-and-out option as a function of underlying price.

Figure 2. shows the value of double barrier call as a function of stock price, but as a comparison the value of plain vanilla call option is also given. It can be seen that double barrier option is cheaper for all asset prices. The time to maturity is T=0.25.

Figures 3. and 4. give option's delta and gamma, respectively. The time to maturity is T=0.25. For asset prices higher that \$107 delta is negative, but it still takes values close to zero. Gamma is negative and takes values close to zero. Until now nothing unusual has happened, but as the maturity is approached, the value of the contract as well as delta and gamma start to behave in a strange manner. For asset prices significantly lower than the upper barrier the call-up-and-out-down-and-out option behaves like plain vanilla call, but close to upper barrier the value of double barrier option suddenly drops, due to very high probability that the barrier is reached and option expires worthless (see Fig. 5.).

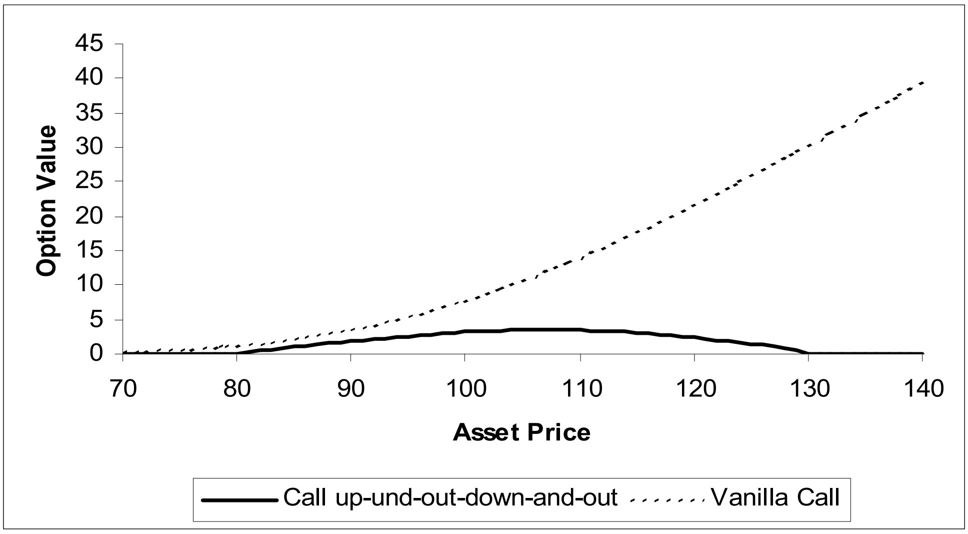

Fig. 2 The value of call-up-and-out-down-and-out and vanilla call options as a function of asset price.

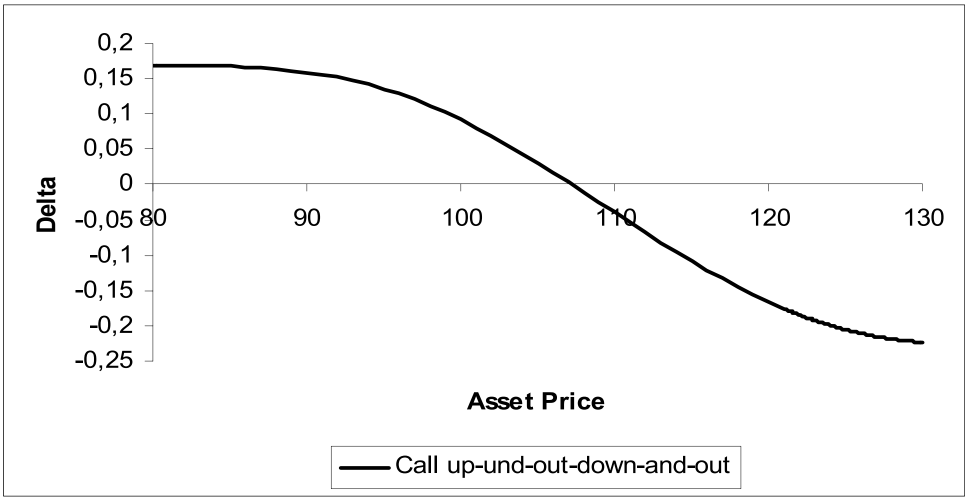

Fig. 3. Delta for call-up-and-out-down-and-out option at T=0.25.

Since option's value decreases, delta becomes negative and takes very low values. The same happens to option's gamma. As mentioned above, this makes the delta hedging very difficult. Firstly, this is due to sudden change of position from short to long (or from long to short; see Fig. 6.) which forces the investor to buy instead of selling (or the opposite). Secondly, delta becomes very unstable close to upper barrier and the position has to be rehedged very often (gamma takes large values, see Fig. 7.).

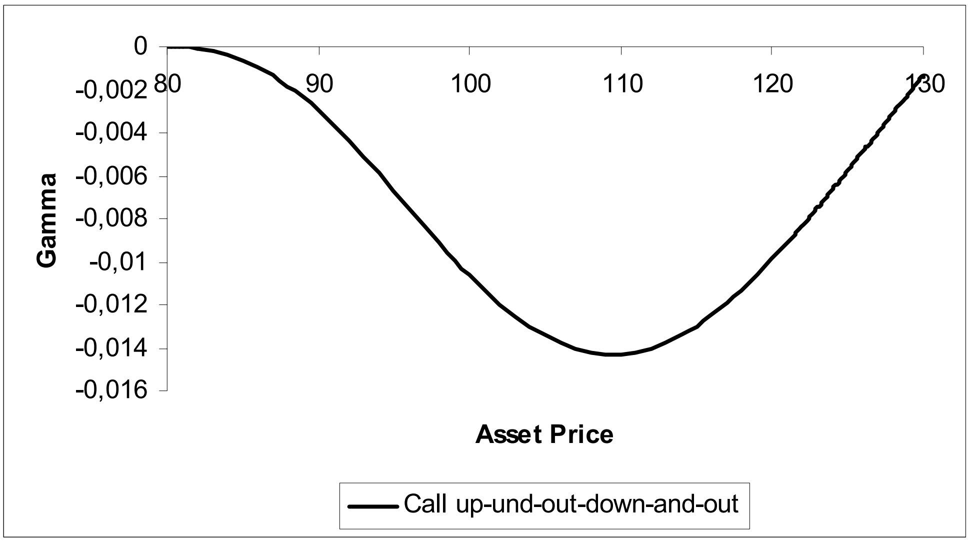

Fig. 4. Gamma for call-up-and-out-down-and-out option at *T*=0.25.

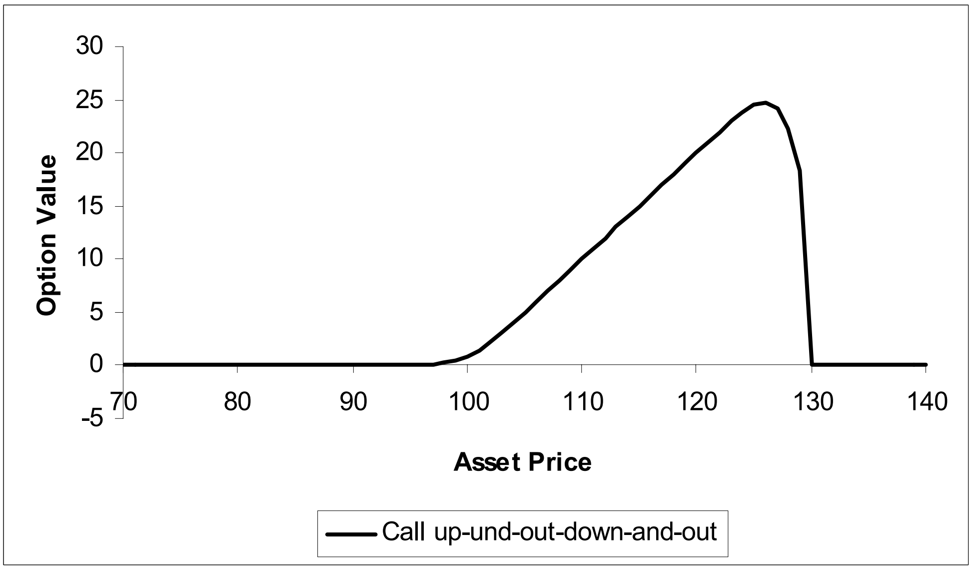

Fig. 5. The value of call-up-and-out-down-and-out option as a function of underlying price. (*T*=1/365)

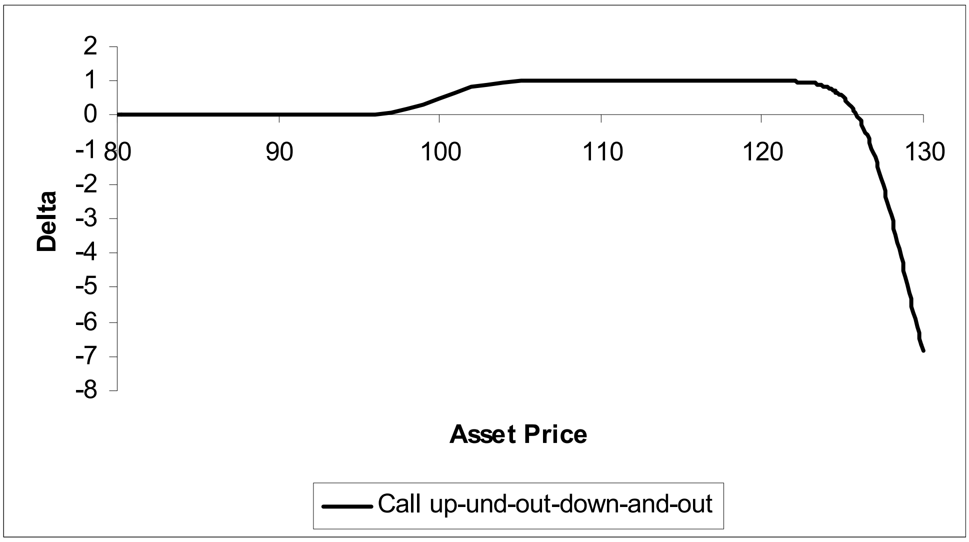

Fig. 6. Delta for call-up-and-out-down-and-out option at *T*=1/365.

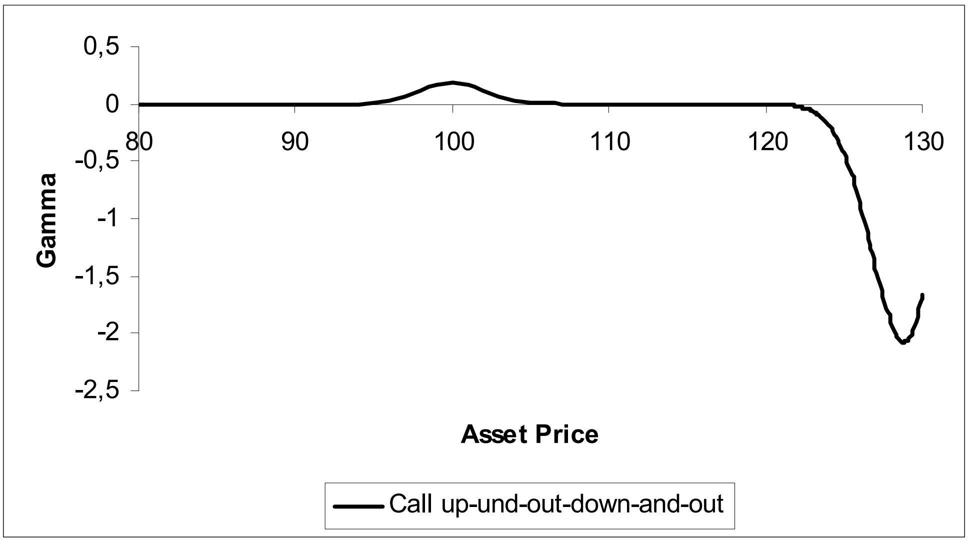

Fig. 7. Gamma for call-up-and-out-down-and-out option at *T*=1/365.

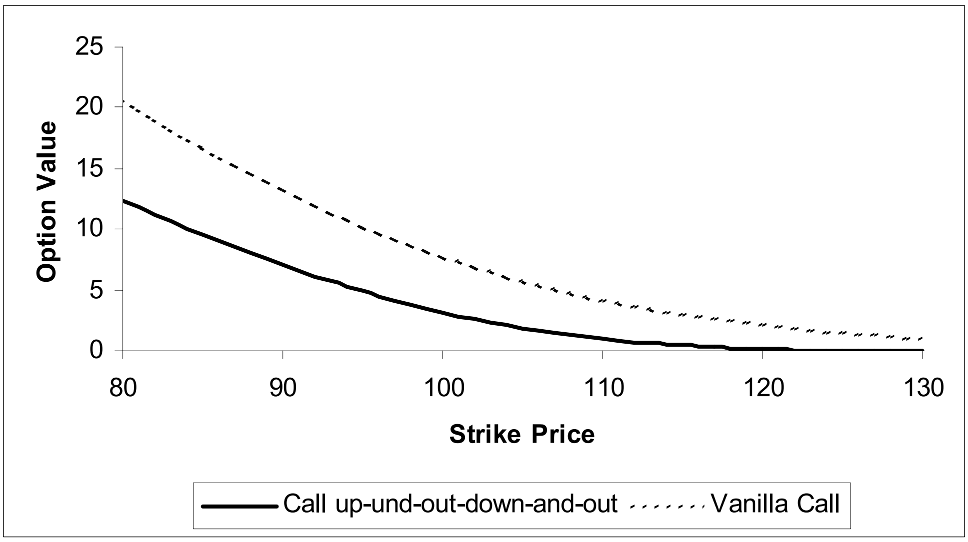

Fig. 8. The value of call-up-and-out-down-and-out and vanilla call options as a function of strike price.

Figure 8. shows value of both call-up-and-out-down-and-out and vanilla call options as a function of strike price between the barriers while underlying spot price remains constant (at \$100). The time to maturity is T=0.25. Again, it can clearly be seen that double barrier option is cheaper when compared to vanilla option.

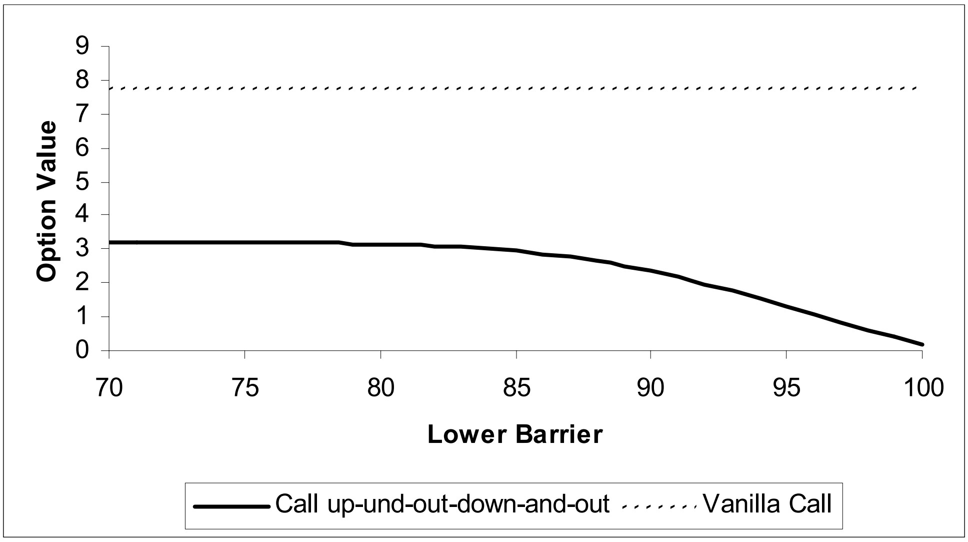

Fig. 9. The value of call-up-and-out-down-and-out and vanilla call options as a function of lower barrier.

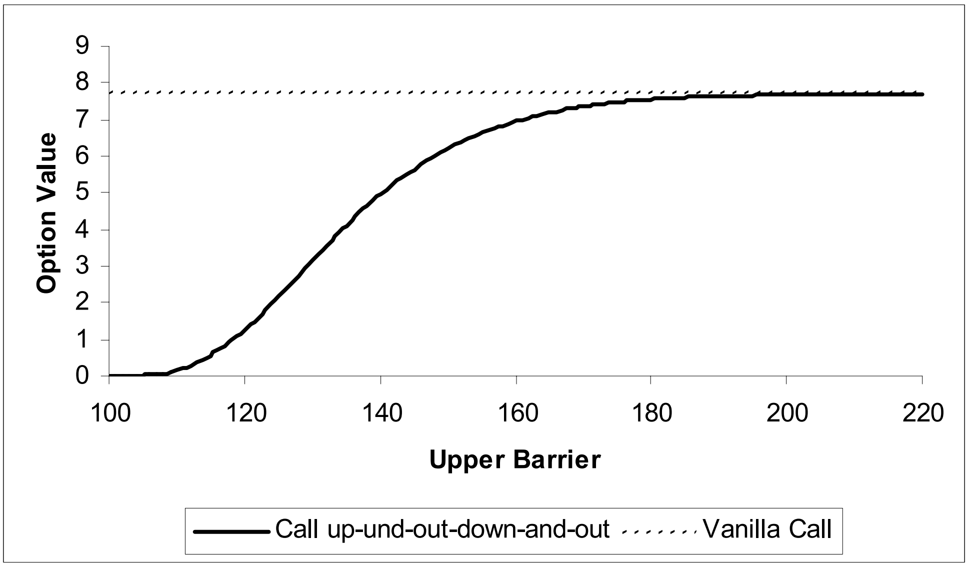

Fig. 10. The value of call-up-and-out-down-and-out and vanilla call options as a function of upper barrier.

Figures 9. and 10. show the value of call-up-and-out-down-and-out contract as a function of lower and upper barrier, respectively. When lower barrier is close to strike price (and current spot price), the value of double barrier option is very close to zero. As the lower barrier decreases, the value of the option increases. For barrier at about \$80 and lower the value of the option stabilizes at about \$3.20. This is due to the fact that the probability that the underlying price reaches the barrier is very low. The value of the call option is determined mainly by the level of upper barrier which has more influence on the payout at option's maturity. As upper barrier increases (with lower barrier constant), the value of double barrier option coincides with the value of vanilla call. On one hand, increased upper barrier means decreased probability of reaching it. On the other hand, the higher the upper barrier, the bigger the potential payout at maturity.

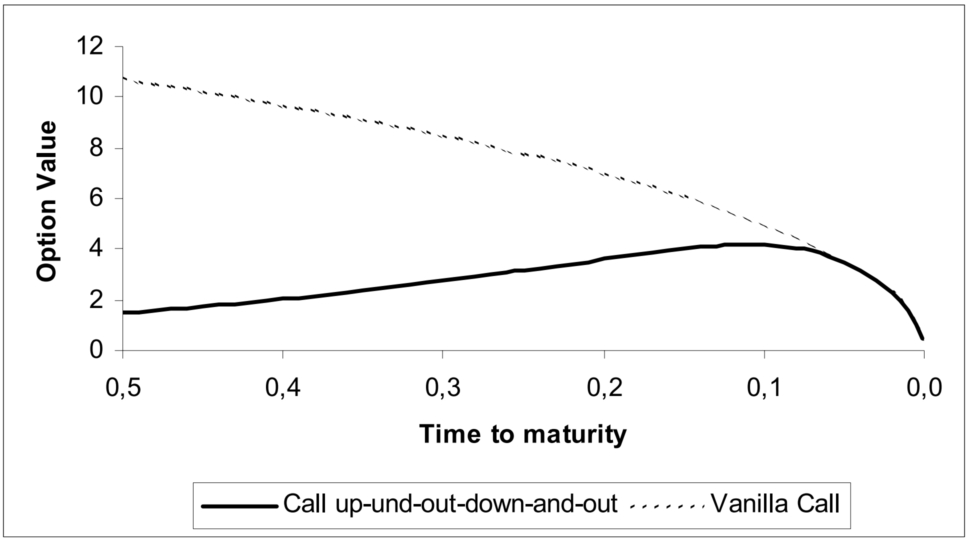

Fig. 11. The value of call-up-and-out-down-and-out and vanilla call options as a function of time to maturity. Both options are at-the-money.

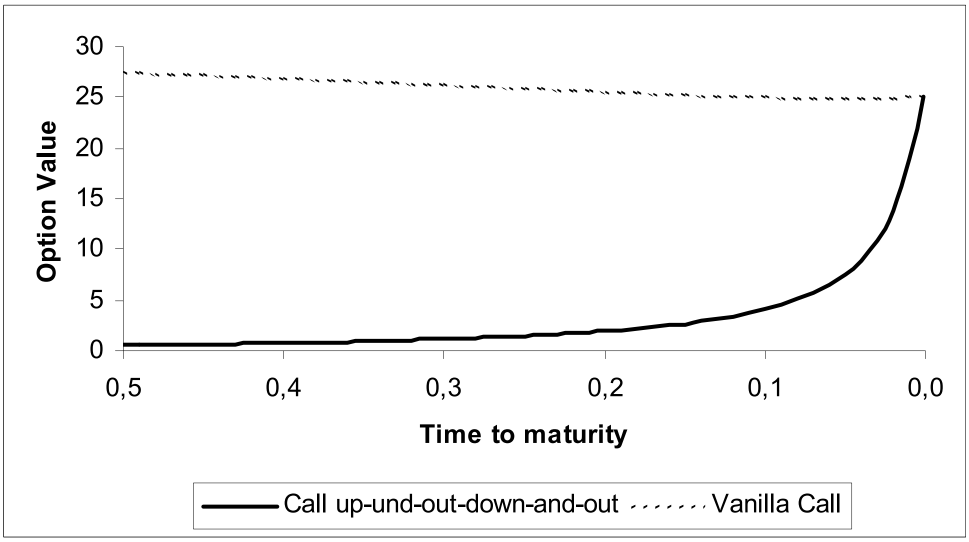

Fig. 12. The value of call-up-and-out-down-and-out and vanilla call options as a function of time to maturity. Both options are in-the-money, underlying price is close to upper barrier.

Figure 11. gives the value of both call-up-and-out-down-and-out and vanilla call options as a function of time to maturity. As both options are at-the-money and long before maturity, double barrier option is much cheaper than vanilla call, but its value grows as the time to maturity decreases (theta is negative), which is never the case for vanilla options. Close to maturity, when the probability of reaching either lower or upper barrier is very low (option is at-the-money), its value coincides with the value of vanilla call and both option expire worthless.

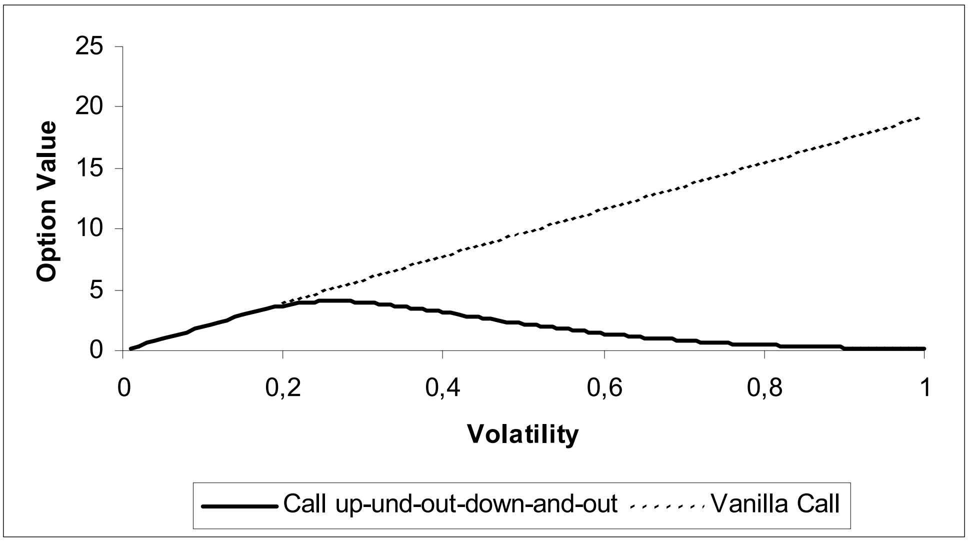

Fig. 13. The value of call-up-and-out-down-and-out and vanilla call options as a function of volatility. Both options are at-the-money.

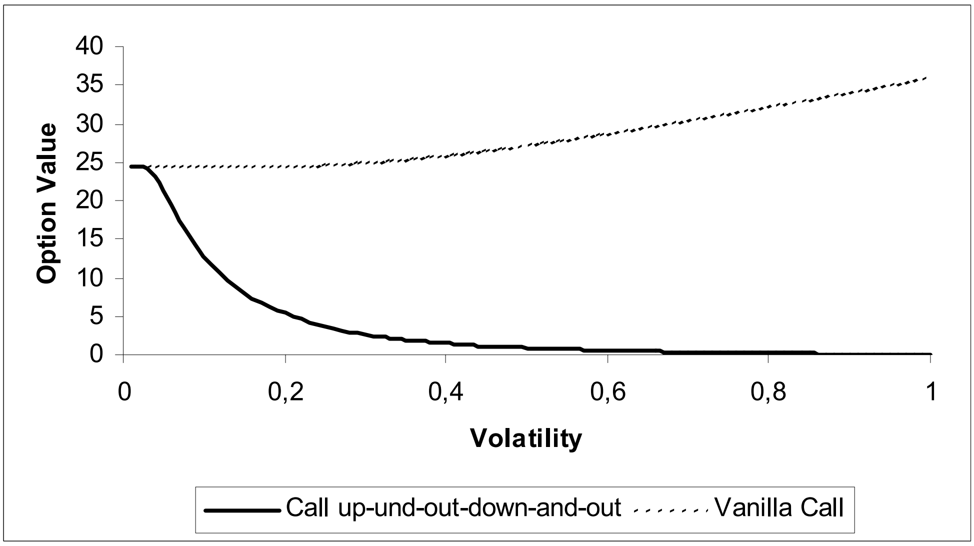

Fig. 14. The value of call-up-and-out-down-and-out and vanilla call options as a function of volatility. Both options are in-the-money, underlying price is close to upper barrier.

Figure 12. shows the value of both call-up-and-out-down-and-out and vanilla call options as a function of time to maturity, but this time both options are in-the money whereas underlying price is close to the upper barrier. Since the probability of reaching upper barrier is very high, the value of double barrier option remains low (when compared to corresponding vanilla call). As time to maturity decreases, the value of double barrier option increases (theta is negative). At maturity both option expire with the same payout.

Figure 13. gives the value of double barrier and vanilla call options as a function of volatility. Since both options are at-the-money and underlying price is relatively far from barriers, for low volatilities both options (double barrier and vanilla) have similar values. As volatility increases, the probability of reaching either lower or upper barrier increases and at some point vega (first derivative of option value with respect to volatility) becomes negative, which is never the case for vanilla options. Similar result can be seen on figure 14. but this time underlying price is very close to upper barrier and double barrier option value decreases for low volatilities as well.

2.6 Summary

Double barrier call-up-and-out-down-and-out option is a sophisticated exotic derivative. It is a tailor-made contract that makes possible to limit investor's exposure to risk. It has two major advantages: low price (when compared to corresponding vanilla option) and flexibility. On the other hand, it has to be handled with care, especially when underlying price approaches upper barrier. Close to upper barrier delta easily becomes negative whereas gamma is negative and large. This makes delta hedging difficult. For large volatilities vega is negative. Finally, for long times to maturity theta is negative.

2.7 References

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3. Asian Call Option

Hervé Fandom Tchomgouo

3.1 Introduction

Asian options also called Average options are securities with a payoff depending on the average value of an underlying stock, index, commodity, foreign currency or interest rates over some time period. The name "Asian" option has no particular significance. David Spaughton tells the story of how both he and Mark Standish were both working for Bankers Trust in 1987. They were in Tokyo on business when they developed the first commercially used pricing formula for options linked to the average price of crude oil. They were in Asia, so they called the options "Asian options." End-users of commodities or energies tend to be exposed to average prices over time, so Asian options are attractive for them. Asian options are also popular with corporations, such as exporters, who have ongoing currency exposures.

Asian options are popular because they tend to be less expensive than comparable vanilla puts or calls. This is because the volatility in the average value of an underlier tends to be lower than the volatility of the value of the underlier itself. Also, in situations where the underlier is thinly traded or there is the potential for its price to be manipulated, an Asian option offers some protection. It is more difficult to manipulate the average value of an underlier over an extended period of time than it is to manipulate it just at the expiration of an option

First introduced in Tokyo[1], Asian Options are among the most popular path-dependent options, since their characteristics capture, in a way, the whole trajectory of the underlying, with a reduced exposure to volatility in most cases. The common belief that these options should be cheaper than their corresponding string of vanilla options is not strictly accurate. However, it happens to often be the case in various practical cases. In addition, Asian options are less sensitive to possible spot manipulations or extreme movements at settlement and offer much flexibility in the way the average is settled. From a trader's point of view, the delta of an Asian option naturally decreases since part of the average becomes known after an observation date. The hedging strategy is therefore eased, compared with regular options. Consequently, Asian options have become very attractive for investors since they provide a customized cheap way to hedge periodic cash-flows. Nonetheless, such options have turned out to be much more difficult to value than standard options.

[1] hence the name of Asian options as opposed to American, European or Bermudean ones.

Previous research was intensively focused on continuous time Asian options using Black-Scholes (1973) assumptions. However, traded Asian options are based on a discrete time sampling and the underlying security can exhibit a pronounced volatility smile as well as non-proportional dividends.

3.2 Definition

Asian options are options in which the underlying variable is the average price over a period of time. Because of this fact, Asian options have a lower volatility and hence rendering them cheaper relative to their European counterparts. They are commonly traded on currencies and commodity products which have low trading volumes. They were originally used in 1987 when Banker's Trust Tokyo office used them for pricing average options on crude oil contracts; and hence the name "Asian" option.

They are broadly segregated into three categories; arithmetic average Asians, geometric average Asians and both these forms can be averaged on a weighted average basis, whereby a given weight is applied to each stock being averaged. This can be useful for attaining an average on a sample with a highly skewed sample population.

To this date, there are no known closed form analytical solutions arithmetic options, due to the property of these options under which the lognormal assumptions collapse. A further breakdown of these options concludes that Asians are either based on the average price of the underlying asset, or alternatively, they are the average strike type.

3.3 Pricing Formulae

To elaborate on arithmetic averaging, this is seen as being the sum of the sampled asset prices divided by the number of samples:

$$Avg_A = \frac{S_1, S_2, \ldots, S_n}{n}$$

and for geometric averaging, the average value is taken as:

$$Avg_G = \sqrt[n]{S_1 S_2 \ldots S_n}$$

where the n^{th} root of the sample values multiplied together is taken.

The payoff functions for Asian options are given as:

For an average price Asian:

$$V = Max(0, \eta(S_A - X))$$

and average strike Asian:

$$V = Max(0, \eta(S_T - S_A))$$

Where η is a binary variable which is set to 1 for a call, and -1 for a put.

Asian's can be either European style or American style exercise.

Here we look some of the models to price standard Asian options under a variety of methods.

3.3.1 Geometric Closed Form (Kemmna & Vorst)

Kemna & Vorst in 1990 put forward a closed form pricing solution to geometric averaging options by altering the volatility and cost of carry term. Geometric averaging options can be priced via a closed form analytic solution because of the reason that the geometric average of the underlying prices follows a lognormal distribution as well, whereas under average rate options, this condition collapses.

The solutions to the geometric averaging Asian call and puts are given as:

$$c_G = Se^{(b-r)(T-t)}N(d_1) - Xe^{-r(T-t)}N(d_2)$$

and,

$$p_G = Xe^{-r(T-t)}N(-d_2) - Se^{(b-r)(T-t)}N(-d_1)$$

Where N(x) is the cumulative normal distribution function of:

$$d_1 = \frac{\ln\left(S/X\right) + (b + 0.5\sigma_A^2)T}{\sigma_A^2\sqrt{T}}$$

$$d_2 = \frac{\ln\left(S/X\right) + (b - 0.5\sigma_A^2)T}{\sigma_A^2\sqrt{T}}$$

which can be simplified to:

$$d_2 = d_1 - \sigma_A^2\sqrt{T}$$

The adjusted volatility and dividend yield are given as:

$$\sigma_A = \frac{\sigma}{\sqrt{3}}$$

$$b = \frac{1}{2}\left(r - D - \frac{\sigma^2}{6}\right)$$

where σ is the observed volatility, r is the risk free rate of interest and D is the dividend yield.

3.3.2 Arithmetic Rate Approximation (Turnbull & Wakeman)

As there are no closed form solutions to arithmetic averages due to the inappropriate use of the lognormal assumption under this form of averaging, a number of approximations have emerged in literature. The approximation suggested by Turnbull and Wakeman (TW) (1991) makes use of the fact that the distribution under arithmetic averaging is approximately lognormal, and they put forward the first and second moments of the average in order to price the option.

The analytical approximations for a call and a put under TW are given as:

$$c_{TW} \approx Se^{(b-r)T_2}N(d_1) - Xe^{-rT_2}N(d_2)$$

$$p_{TW} \approx Xe^{-rT_2}N(-d_2) - Se^{(D-r)T_2}N(-d_1)$$

where:

$$d_1 = \frac{\ln\left(S/X\right) + (b + 0.5\sigma_A^2)T_2}{\sigma_A\sqrt{T_2}}$$

$$d_2 = d1 - \sigma_A\sqrt{T_2}$$

Where T_2 is the time remaining until maturity. For averaging options which have already begun their averaging period, then T_2 is simply T (the original time to maturity), if the averaging period has not yet begun, then T_2 is $T_2 - \tau$.

The adjusted volatility and dividend yield are given as:

$$\sigma_A = \sqrt{\frac{\ln(M_2)}{T} - 2b}$$

$$b = \frac{\ln(M_1)}{T}$$

To generalise the equations, we assume that the averaging period has not yet begun and give the first and second moments as:

$$M_1 = \frac{e^{(r-D)T} - e^{(r-D)\tau}}{(r-D)(T-\tau)}$$

$$M_2 = \frac{2e^{(2(r-D)+\sigma^2)T}S^2}{(r-D+\sigma^2)(2r-2q+\sigma^2)T^2} + \frac{2S^2}{(r-D)T^2}\left(\frac{1}{2(r-D)+\sigma^2} - \frac{e^{(r-D)T}}{r-D+\sigma^2}\right)$$

If the averaging period has already begun, we must adjust the strike price accordingly as:

$$X_A = \frac{T}{T_2}X - \frac{(T-T_2)}{T_2}S_{Avg}$$

Where we reiterate T as the original time to maturity, T_2 as the remaining time to maturity, X as the original strike price and S_{Avg} is the average asset price. Haug (1998) notes that if $r=D$, the formula will not generate a solution.

3.3.3 Arithmetic Rate Approximation (Levy)

Levy puts forward another analytical approximation which is suggested to give more accurate results than the TW approximation. We look at the differences later.
The approximation to a call is given as:

$$c_{Levy} \approx S_Z N(d_1) - X_Z e^{-rT_2} N(d_2)$$

and through put-call parity, we get the price of a put as:

$$p_{Levy} \approx c_{Levy} - S_Z + X_Z e^{-rT_2}$$

Where

$$d_1 = \frac{1}{\sqrt{K}}\left[\frac{\ln(L)}{2} - \ln(X_Z)\right]$$

$$d_2 = d_1 - \sqrt{K}$$

and

$$S_Z = \frac{S}{(r-D)T}\left(e^{-DT_2} - e^{-rT_2}\right)$$

$$X_Z = X - S_{Avg}\frac{T-T_2}{T}$$

$$K = \ln(L) - 2[rT_2 + \ln(S_Z)]$$

$$L = \frac{M}{T^2}$$

$$M = \frac{2S^2}{r-D+\sigma^2}\left\{\frac{e^{(2(r-D)+\sigma^2)T_2-1}}{2(r-D)+\sigma^2}\right\} - \frac{e^{(r-D)T_2}-1}{r-D}$$

where the variables are the same as defined under the TW approximation.

Furthermore, transposing the 2 call values as a function of the strike price illustrates the similarity between the two methods.

3.3.4 Arithmetic Rate Approximation (Curran)

Curran (1992) gives an approximation based on a geometric conditioning approach.

3.3.5 Arithmetic Rate Approximation (Monte Carlo Simulation)

Various methods using Monte Carlo simulation (MCS) have been developed to price arithmetic Asian options. The aforementioned analytical approximations by TW, Levy and Curran can all be computed using a simulation method. Monte Carlo simulation can give relatively accurate prices for option values, and in the case of Asian options, which are highly path dependent, this method is particularly useful.

In section Geometric Closed Form (Kemna & Vorst), we gave a geometric closed form solution to Asian options originally presented by Kemna & Vorst (1990). The authors show that the solution to the geometric solution can be used as a control variate within a MCS framework.

The control variate technique can be used to find more accurate analytical solutions to a derivative price if there is a similar derivative with a known analytic solution. With this in mind, MCS is then undertaken on the two derivatives in parallel.

Given the price of the geometric Asian, we can price the arithmetic Asian by considering the equation:

$$V_A = V_A^* - V_B^* + V_B$$

Where V_A^* is the estimated value of the arithmetic Asian through simulation, V_B^* is the simulated value of the geometric Asian, and V_B is the exact value of the geometric Asian given above.

3.4 Characteristics Of Asian Options

- Although other type of averages (such as geometric average) is possible, almost all traded average options use equally weighted arithmetic averages,
- No continuous averaging is used,
- Average price is more common than average strike,
- European is more common than American,
- For an at the money option, it is roughly half of the corresponding European option,
- Volatility of average is smaller than volatility of final stock price.

3.5 Pro & Cons

Asian options are similar to standard financial options in that they provide protection against markets and liquidity risks. However, with an asian option, the market index is averaged over an agreed time period. This means that the pay-off is the difference between the average indices and the strike price. The advantages are:

- Lower volatility,
- Matching actual regular exposures,
- Resolving settlement uncertainty,
- When the option is close to maturity, almost all stock prices in the average has been realized, so not too worry about the final price,
- Delta is gradually decreasing, making replication easier,
- For an at the money option, it is roughly half of the corresponding European option,
- Volatility of average is smaller than Volatility of final stock price.

Although Asian options are cheaper to trade, they have some limitations. These are:

- Lower option price,
- Although other type of averages (such as geometric average) is possible, almost all traded average options use equally weighted arithmetic averages,
- No continuous averaging is used,
- Average price is more common than average strike,
- European is more common than American,
- The lognormal assumptions collapse to find analytical solutions to arithmetic options.

3.6 Comparison of an Asian Option and a Vanilla Call

We have collected the spot prices of copper from The London Metal Exchange. The Turnbull and Wakeman arithmetic average approximation has been used to compute the price of Asian options. We have two different scenarios which represent a bullish and a bearish market. On every figure shown below, we will compare the value of Asian options and vanilla call. Later we will show the changes of asian option value with respect to strike price, volatility and spot price within the averaging period. The volatility has been astimated from copper spot prices data and is equal to 17%.

First scenario: A bullish market

The copper prices have been gathered from The Londom Metal Exchange for a period of 127 days from March 31, 2003 to September 30, 2003.

Table 1. Valuation of Asian Option and Vanilla Call.

	Asian Call	Vanilla Call
Spot price (S)	\$1623,00	\$1623,00
Average price (S_{AV})	\$1623,00	-
Strike price (X)	\$1625,00	\$1625,00
Time to start of average period (t)	0,00	-
Original time to maturity (T)	0,50	0,50
Remaining time to maturity (T_2)	0,50	-
Risk-free rate (r)	5,00%	5,00%
Cost of carry (b)	2,00%	2,00%
Volatility (σ)	17,00%	17,00%
Value	**\$45,7980**	**\$81,3816**

The value of Asian call and vanilla call have been computed using copper prices. In this case, we are inn a bullish market.

Figure 1 shows the variations of copper prices over time. The average price is growing smoothly when compared to spot price which is growing with larger ups and downs. The average price of copper ensure to the trader of such a metal, an average buying price (for a final user) or an average selling price (for a seller). The buyer of copper is able to hedge its position on copper market using Asian call option. A drastic increase of copper spot price will not affect the average price much. An increase in copper price will not affect the profit margin of copper buying company, because its position is protected by Asian option.

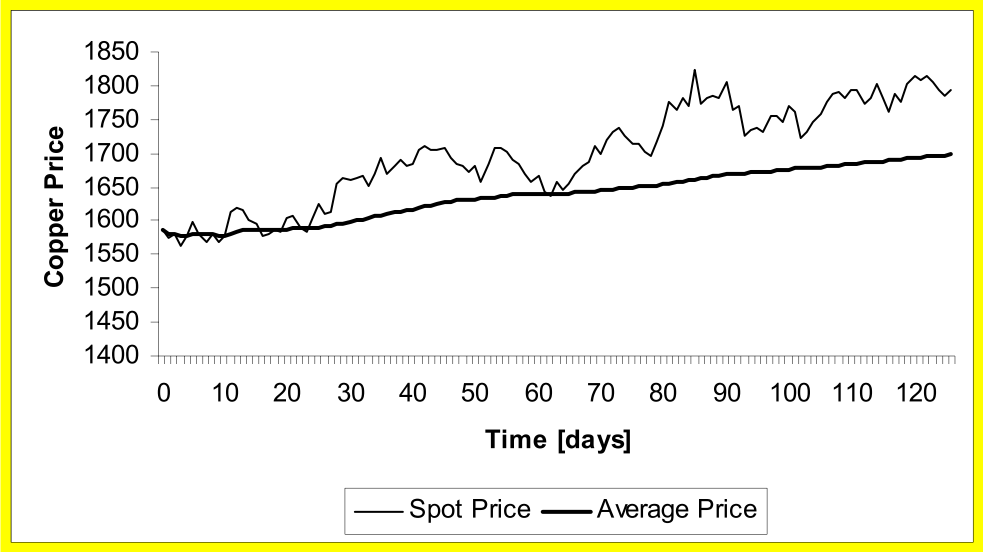

Fig. 1. Copper price in dollars per ton.

Figure 2 shows the option values of Asian call and vanilla call over a period of 127 days. The option value of Asian option varies with a smaller deviation (volatility) than Vanilla call. This option price is also lower than Vanilla one, that makes it cheaper to trade. Close to maturity the value of Asian option remains stable when compared to the value of Vanilla call. This ensures the stability of payout of Asian option even on very shallow markets.

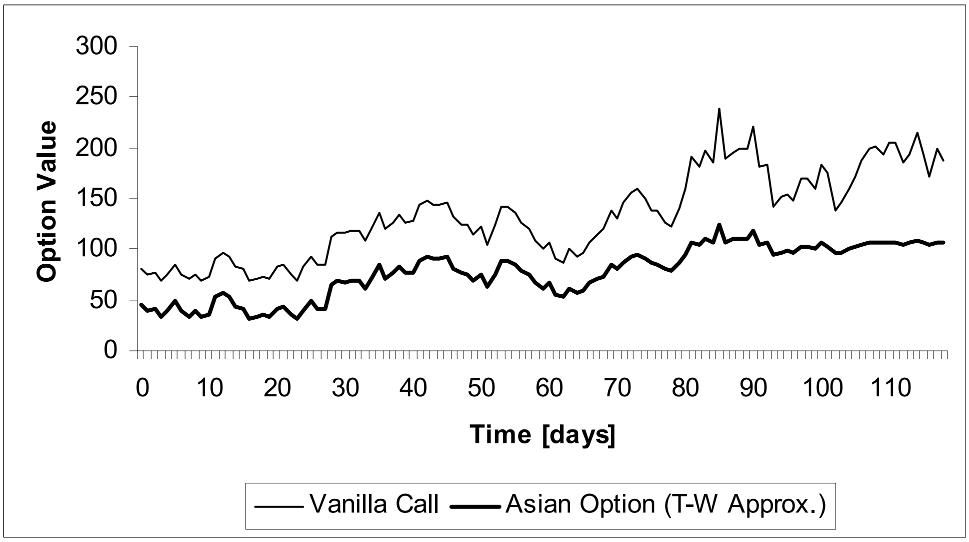

Fig. 2. Variation of option value with respect of time.

Second scenario: A bearish market

The copper prices have being gathered from The Londom Metal Exchange for a period of 127 days as from march 28, 2002 to September 30, 2002.

Table 2. Valuation of Asian option and Vanilla Call.

	Asian Call	Vanilla Call
Spot price (S)	\$1623,00	\$1623,00
Average price (S_{AV})	\$1623,00	-
Strike price (X)	\$1625,00	\$1625,00
Time to start of average period (t)	0,00	-
Original time to maturity (T)	0,50	0,50
Remaining time to maturity (T_2)	0,50	-
Risk-free rate (r)	5,00%	5,00%
Cost of carry (b)	2,00%	2,00%
Volatility (σ)	17,00%	17,00%
Value	**\$47,0969**	**\$83,4751**

The value of Asian call and Vanilla call have been computed using copper prices. In this case, we are in a bearish market.

Figure 3 shows the variations of copper prices over time. The average price is declining slowly and smoothly at the end of the period as compare to spot price which is decling faster and with large ups and downs. The average price of copper ensures to the trader of such a metal, an average buying price (for a final user) or an average selling price (for a seller). In this scenario both Asian call and Vanilla call are out-of-the-money.

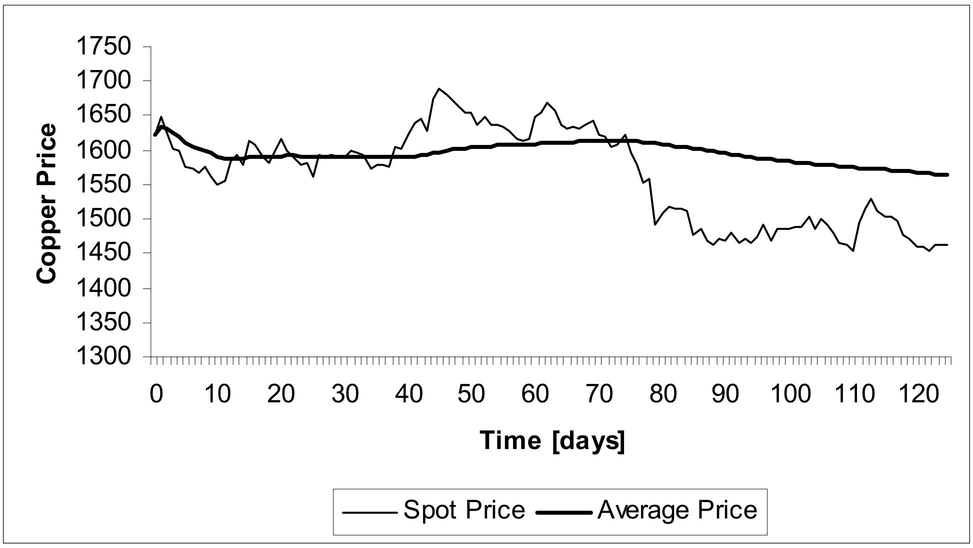

Fig. 3. Copper price in dollars per ton.

Figure 4 shows the option values of Asian and Vanilla call over a period of 127 days. The option value of Asian call varies with a smaller deviation (volatility) than Vanilla call. This option price is also lower than Vanilla one, that makes it cheaper to trade.

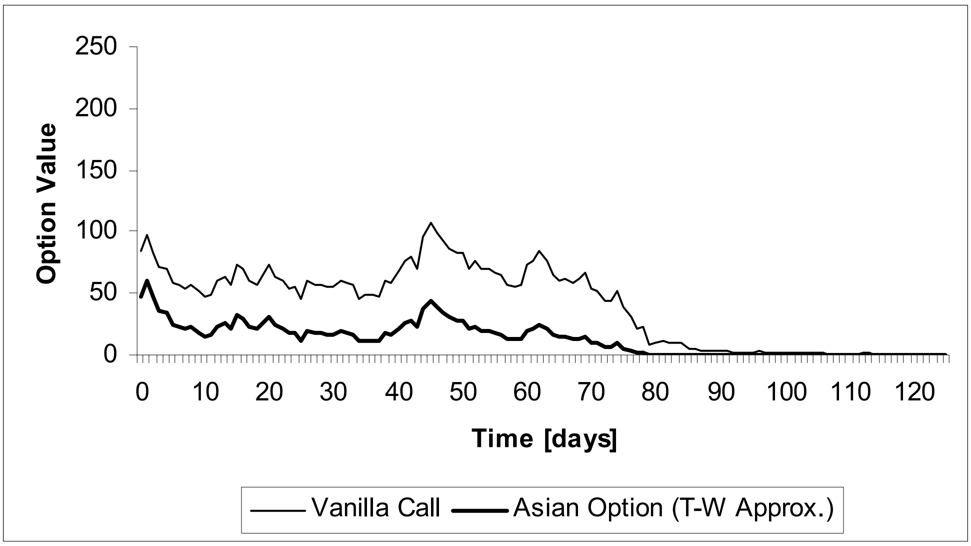

Fig. 4. Variation of option value with respect of time.

Figure 5 show the Asian option with respect to strike price. Option value decreases as strike price increases. The Asian Option curve is under that of a Vanilla call. This means that Asian option is cheaper than Vanilla call.

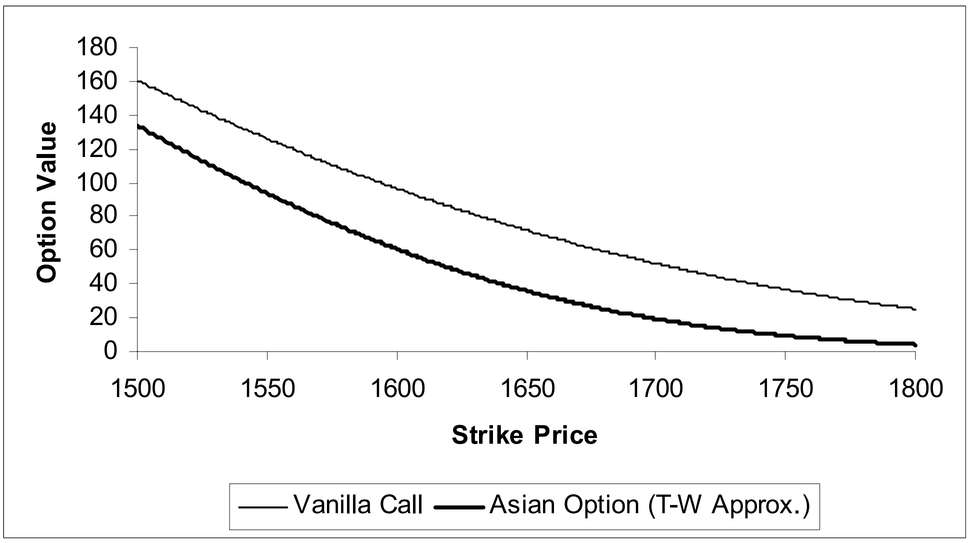

Fig. 5. Variation of Option value subject to change in Strike Price.

Figure 6 shows how a continuous increase of one point in volatility affects the option value. This means that Asian option is cheaper than Vanilla call.

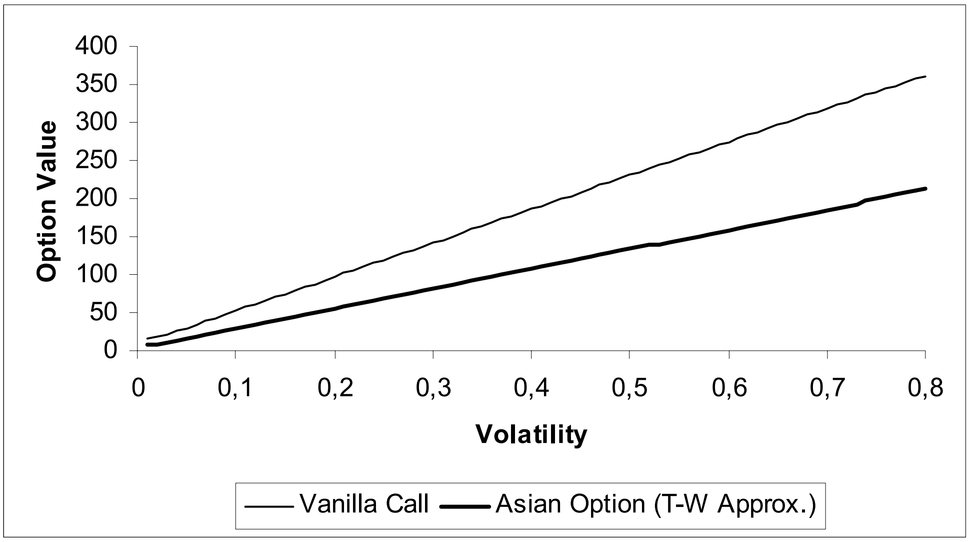

Fig. 6. Variation of Option value subject to an increase in Volatility.

On Figure 7 we see how changes in spot price affect option value. Option Value increases with respect to an increase in the spot price.

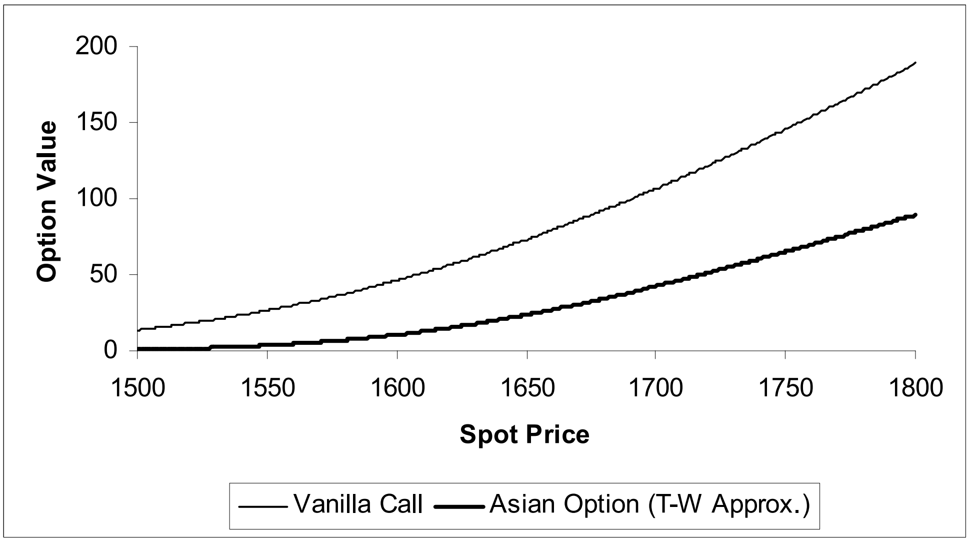

Fig. 7. Variation of Option value subject to change in the spot price.

Figure 8 shows the Delta as a function of spot price. The delta of an Asian option is naturally lower since part of the average becomes known after an observation date.

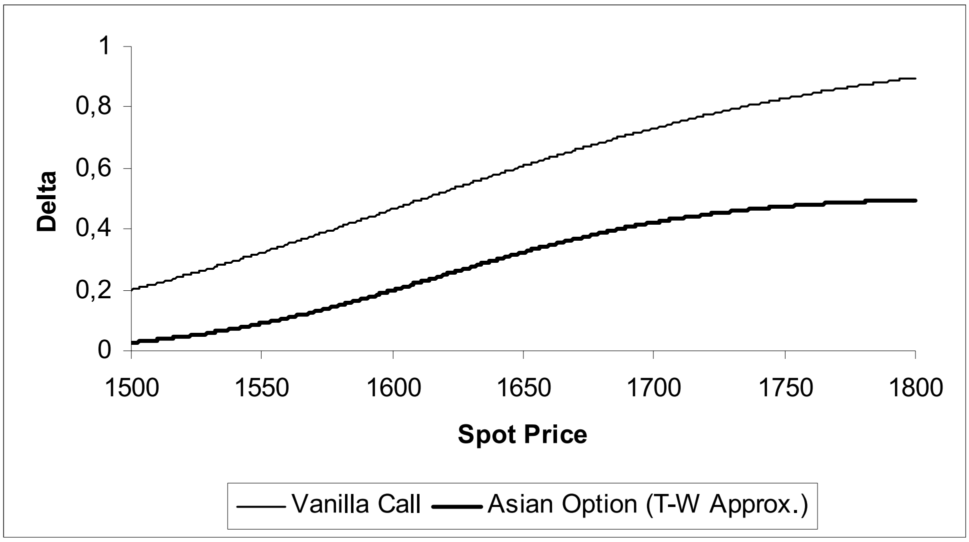

Fig. 8. Variation of Delta subject to change in the Spot Price.

3.8 Applications Of Asian Options

Asian options have a broad range of applications. Companies which deal with such an option are the ones that are using or trading commodities with fluctuating prices. When they purchase or sell their product (Crude Oil, Minerals, etc…), they purchase Asian Options to protect themselves against prices rises or fall respectively. This protection gives them the opportunity to maintain an average buying or selling price no matter the Commodity Market Prices fluctuations.

3.9 Summary And Conclusion

Asian Options are commonly traded on currencies and commodity products which have low trading volumes. They are options that ensure to its buyer an average return at a lesser risk as compare to Vanilla Call.

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4. Interest Rate Collar

Francisco J. Gil

4.1 Definition

A collar is a combination of a Cap and a Floor. First of all we are going to see the definitions of a Cap and a Floor.

4.1.1 What is a Cap?

An option contract which puts an upper limit on a floating exchange rate. The writer of the Cap has to pay the holder of the Cap the difference between the floating rate (which is determined by free market forces) and the reference rate when that reference rate is breached. There is a premium to be paid by the buyer of such a contract in order to gain the certainty of a maximum payout.

An interest rate Cap consists of a series of individual European call options, called **caplets.**

A series of Caplets, or Cap can extend for up to 10 years in most markets. Caps are also known as Ceilings.

4.1.2 What is a Floor?

An Interest Rate Floor is a contract that guarantees a minimum level of interest rate. A Floor can be guaranteed for one particular date. In return for making this guarantee, the buyer pays a Premium.

4.2 Pricing Formulae

The price of a Cap is the sum of the price of the caplets that make up the Cap. Similarly, the value of a floor is the sum of the sequence of individual put options, often called floorlets ,that make up the floor.

$$Cap = \sum_{i=1}^{n} Caplet_i \ , \ Floor = \sum_{i=1}^{n} Floorlet_i$$

where

$$Capletvalue = \frac{Notional * \frac{d}{Basis}}{(1 + f * \frac{d}{Basis})} * e^{-rT} * \left[FN(d_1) - XN(d_2)\right]$$

d is the numbers of days in the forward rate period.

Basis is the day basis o number of days per year used in the market (i.e. 360 or 365)

$$Floorlet\ value = \frac{Notional * \frac{d}{Basis}}{(1 + f * \frac{d}{Basis})} * e^{-rT} * \left[XN(-d_2) - FN(-d_1) \right]$$

where

$$d_1 = \frac{\ln(F/X) + (\sigma^2/2)T}{\sigma\sqrt{T}}$$

$$d_1 = \frac{\ln(F/X) - (\sigma^2/2)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$$

The Collar protects from the risk of rising and falling interest rates. The purchase of a collar comprises the simultaneous purchase of a cap and a sale of a Floor with identical maturities, national principals and reference rates. This is a common strategy for reducing the cost of the premium to insure against an adverse movement in short term interest rates. The premium obtained from the sale of the floor will in most cases only partially offset the cost of the cap. This cost can be reduced by raising the strike of the floor, when the premium of the floor exactly matches that of the cap this is known as a **Zero Cost Collar**. The sale of a collar works vice versa.

The buyer of a collar wants to hedge against rising interest rates and lowers his hedging costs through the sale of a floor. Therefore, he benefits from possible declines in interest rates only down to the floor.

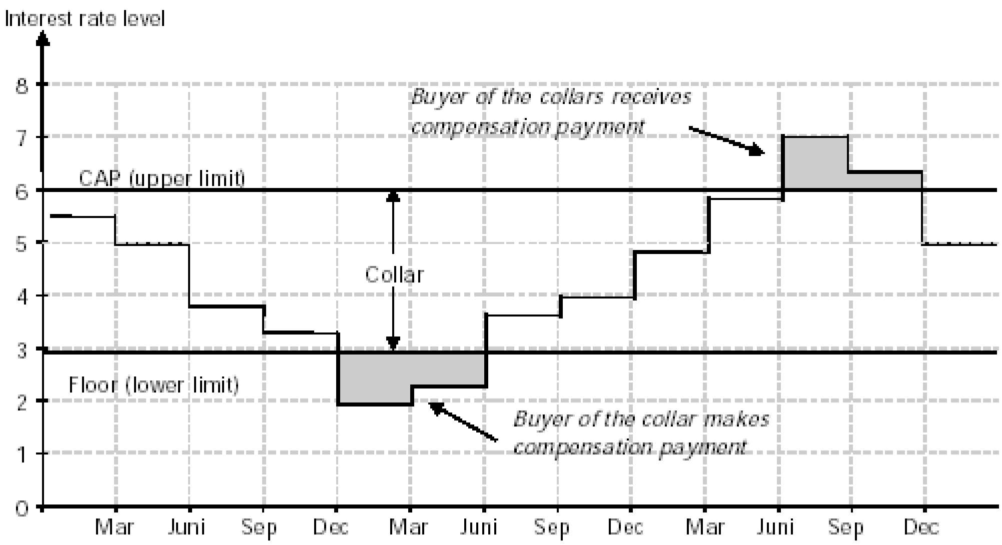

Fig. 15. Interest rate fluctuations and interest rate collar.

If the reference rate is lower than the floor, the buyer of the collar is obliged to pay the seller the difference of the collar. A collar effectively takes care of this problem by separating the exercise prices of the cap and floor. Consider a borrower who buys a cap with a strike of 6%. This party pays a premium up front and is protected against increases in interest rates above 6%. Suppose the party sells a floor with an exercise rate of 3%. The counterparty, who is the buyer of the floor, receives the benefit of decreases in interest rates below 3%. Thus, the borrower will have to pay when interest rates are below 3%. For the willingness to pay when rates are below 3%, the borrower receives a premium. This premium can be used to partially or fully offset the premium on the cap. For a given exercise rate on the cap, there is a unique exercise rate on the floor that will produce a premium received on the floor that offsets the premium paid on the cap. Trial and error with different exercise prices has to be used to determine the exercise rate on the floor that produces a floor premium that offsets the cap premium. Of course it is not necessary that the cap and floor premiums offset. If the floor premium is less than the cap premium, the buyer of the collar would have to pay some amount of money up front. If the floor premium exceeds the cap premium, the buyer of the collar would actually receive some money up front.

4.3 Features

- The collar reduces the cost of interest-rate protection. This is the aim of a collar.
- Banks provide Interest-Rate Collars in all other major currencies.
- Banks can arrange different maturities.
- Interest-Rate Collars are generally set against LIBOR but banks can set them against any other recognized rate.
- Banks usually pay, or ask the firm to pay, compensation at the end of each relevant LIBOR period.
- The collar provides protection against higher interest rates.
- The firm can sell the collar back to the bank at any time.
- The firm can pay for the cost of an Interest-Rate Collar up front or over the life of the deal.
- If the firm has a zero-cost interest-rate collar, the firm do not have to pay any premiums.
- The main disadvantage of a collar is that the firm has to pay a certain minimum rate of interest and the firm loses some of the possible benefit of lower interest rates.

4.4 Who uses Interest Rate Collars?

Variable rate borrowers are typical users of Interest Rate Collars. They use Collars to obtain certainty for their borrowings by setting the minimum and maximum interest rate they will pay on their borrowings. By implementing this type of financial management, variable rate borrowers obtain peace of mind from the knowledge that interest rate changes will not impact greatly on the borrowing costs, with the resultant freedom to concentrate on other aspects of their business.

4.5 How does an Interest Rate Collar work?

An Interest Rate Collar ensures that you will not pay any more than a pre-determined level of interest on your borrowings (the interest rate will be between a range).

The buyer and the seller agree upon the term, the cap and floor strike rates, the notional amount (usually set equal to borrowed amount), the amortization ("bullet", mortgage, straight line ,etc..), the start date, and the settlement frequency. If at any time during the tenor of the collar, the interest rate moves above the cap strike rate or below the floor strike rate, one party will owe the other a payment. The payment is calculated as the difference between the strike rate and the underlying interest rate times the notional amount outstanding times the day's basis for the settlement period.

An Interest Rate Collar enables variable rate borrowers to retain the advantages of their variable rate facility while obtaining the additional benefits of a maximum interest rate, at a reduced cost to an Interest Rate Cap.

4.6 Are there any risks associated with an Interest Rate Collar?

There are risks associated with an Interest Rate Collar. It is important to understand that if interest rates fall below the Floor rate, you will have missed out on the potential reduction to your cost of funds. The cost advantages over an Interest Rate Floor may or may not compensate for this potential loss. Only you can decide if the premium savings outweigh the potential of reduced cost in a falling interest rate environment.

4.7 How much does an Interest Rate Collar cost?

The cost of the Collar is referred to as the premium. The premium for an Interest Rate Collar depends on the rate parameters you want to achieve when compared to current market interest rates.

4.8 Example

Table 3. Yield to maturity for zero-coupon bonds and specified dates to maturity.

Date reset in:	Yield to maturity:
180	2.00%
360	2.50%
540	2.90%
720	3.15%

Table 1 gives yield to maturity for zero-coupon bonds for specified maturities. These interest rates will be used to calculate 180-day forward rates which are necessary to calculate the value of the cap and the floor. In this example the volatility of forward rate is assumed to be 5% per annum. The notional principle is $1,000,000.

The formula used to calculate forward rates is:

$$f_{t_2-t_1} = \left[\frac{(1+s_{t_2})^{t_2}}{(1+s_{t_1})^{t_1}}\right]^{1/(t_1-t_2)} - 1$$

where s is a spot rate and f is a forward rate.

Table 4. 180-day forward rates for given maturities.

Forward rate period	Forward rate:
0-180	2.00%
180-360	2.97%
360-540	3.26%
540-720	3.35%

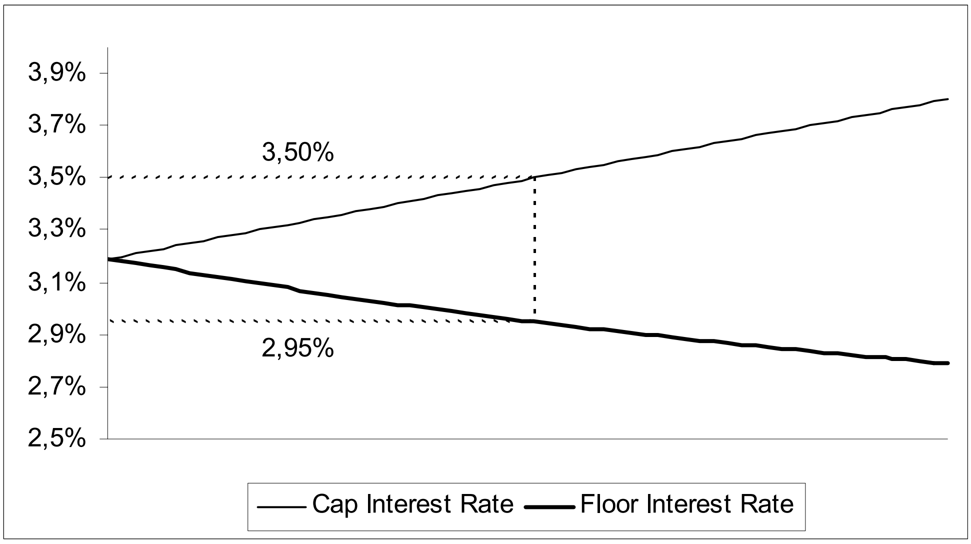

Fig. 16. The relation between cap and floor strike rates.

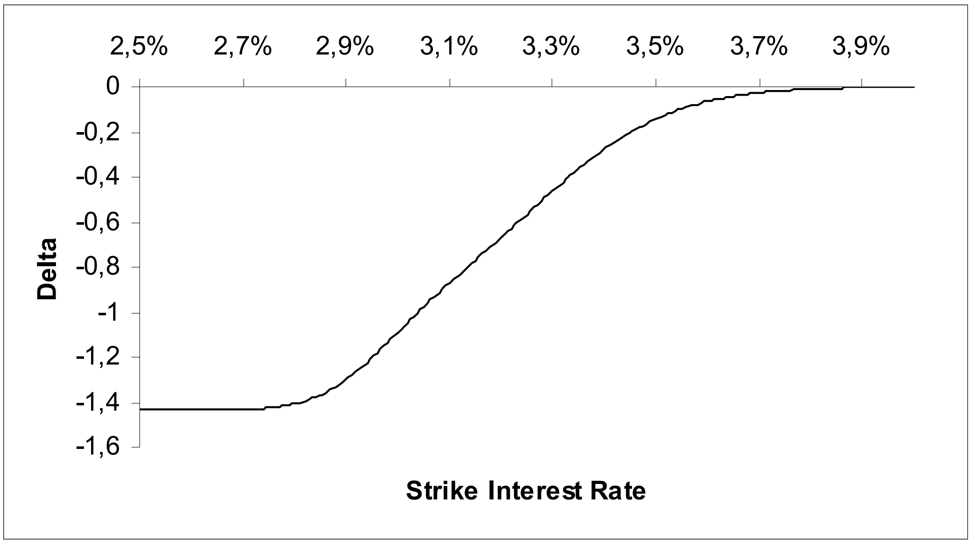

Fig. 17. Delta of interest rate cap.

In Figure 16 we can see the relation between the cap and the floor interest rate to hedge against an increase of the interest rate. It means that if we buy a cap with an interest rate of 3,50% , we have to sell a floor with an interest rate of 2,95% to offset the premium of the cap.

Below 2,7% delta reaches its minimum value and is constant. Between 2.7% and 3.9% delta increases. Above 3.9% delta remains constant at zero.

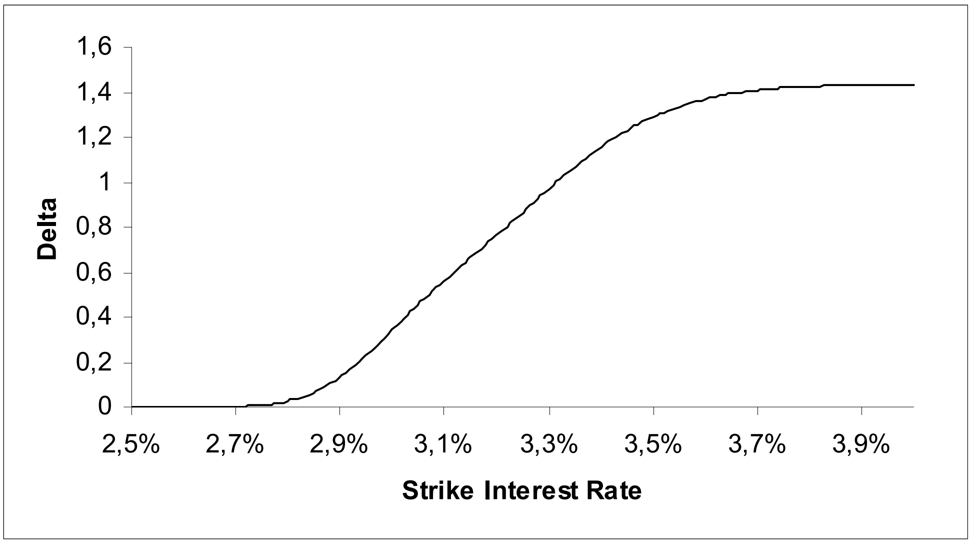

Fig. 18. Delta of interest rate floor.

We can see the delta is positive while in above plot it is negative. Below 2.7% it is zero and above 3.9% is constant at about 1.4.

4.9 Conclusions

As we have seen a collar interest rate is a combination between a cap and a floor, this is very useful when we want protect ourselves against the increase of the interest rate.

We have got a premium reduction due to the fact that we earn money by writing the floor. It means that buy a collar is cheaper than buy only a cap, although it is more risky, because you have to pay a minimum interest rate. The best way to reduced the cost of collar has been shown in the above example (Figure 16).

The Interest Rate collars fluctuation depend on the LIBOR or another rate recognized.

The variable rate borrowers are users of collars. It is a way to make sure themselves against losses.

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Table of Figures